

CELLULAR AUTOMATA ON A  $G$ -SET

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**ABSTRACT.** In this paper, we extend the usual definition of cellular automaton on a group in order to deal with a new kind of cellular automata, like cellular automata in the hyperbolic plane and we explore some properties of these cellular automata. This definition also allows to deal with maps, intuitively considered as cellular automata, even if they did not match the usual definition, like the Margolus billiard-ball. One of the main results is an extension of Hedlund's theorem for these cellular automata.

## 1. INTRODUCTION

Cellular automata have been developed first by John von Neumann [11] on an infinite rectangular grid. Originally, the cells were the squares of an infinite 2-dimensional checker board, addressed by  $\mathbb{Z}^2$ . Later it had been extended to a  $d$ -dimensional board, addressed by  $\mathbb{Z}^d$  (see e.g. [6]). In modern cellular automaton theory, the lattice structure is provided by any group  $G$  (see e.g. [2]). This latter case shall be referred to as the *classical case* in the rest of the present paper. Ever since, cellular automata have been used in various topics like group theory, but also language recognition, decidability questions, computational universality, dynamical systems, conservation laws in physics, reversibility in microscopic physical systems.

Recently cellular automata have been developed in a new environment by Margenstern and Morita [9]: the grid is provided by a tessellation of the hyperbolic plane  $\mathbb{H}^2$ . Let's recall the theorem of Poincaré: the Coxeter group of a tessellation (i.e., the group generated by reflections with respect to the sides of the polygons of the tessellation) acts freely on the tessellation if every angle of the polygons of the tessellation is  $\frac{2\pi}{p}$  for some even number  $p$ . The classical case may be useless in this context if the hypothesis of Poincaré's theorem is not verified and then there is no natural group addressing the tiles. Yet, there are groups acting on the tiles like the group of isometries of  $\mathbb{H}^2$  preserving the tessellation or the Coxeter group. Margenstern [7, 8] obtained good results on this new kind of cellular automata in the specific context of a regular tessellation of the hyperbolic plane. But this extension of the definition of a cellular automata has not been investigated yet on a theoretical aspect. Hence this paper defines and studies what is a cellular automaton defined on a set equipped with a group action, also called a  $G$ -set. The only requirement is the transitivity of the action. This condition is essential since the local definition of a cellular automaton has to be propagated on the whole set. The results of this paper may be applied to the tessellation of the hyperbolic plane, but also to any tiling in higher dimensional hyperbolic spaces or even to unusual tiling of any Euclidean space.

Section 2 defines what a coordinate system is, i.e., a choice of addressing the cells. Section 3 defines what is a cellular automaton on a set equipped with a transitive group action. Section 4 defines what equivariant cellular automata are. This class of cellular automata is the one which have the most similarity with the ones of the classical case. Section 5 investigates the properties of the memory set of cellular automata, and how they are related to the coordinate systems. It

will be proved that there is only one minimal memory set, up to the origin of the coordinate system. In Section 6, we give a characterization of equivariant cellular automata which is an analogue of Hedlund's theorem. Section 7 studies the stability of the composition of cellular automata. In particular, there exist cellular automata which, when composed with themselves, are no longer cellular automata.

We would like to express our gratitude to Maurice Margenstern for inspiration, motivation and good discussions. We are also greatly thankful to Tullio Ceccherini-Silberstein and Michel Coornaert for their support and numerous suggestions and remarks.

## 2. COORDINATE SYSTEM

Let  $\Gamma$  be a set equipped with a transitive left action of a group  $G$ . For  $\alpha \in \Gamma$ , let  $\text{Stab}(\alpha) = \{g \in G : g \cdot \alpha = \alpha\}$  denote the stabilizer subgroup of  $\alpha$  in  $G$ . As we have

$$\text{Stab}(g \cdot \alpha) = g\text{Stab}(\alpha)g^{-1}$$

for all  $g \in G$ , all the stabilizer subgroups are conjugate since the action is transitive.

Consider the set  $G/\text{Stab}(\alpha) = \{g\text{Stab}(\alpha) : g \in G\}$  of the left cosets of  $\text{Stab}(\alpha)$  in  $G$ . A subset  $T \subset G$  is a complete system of representatives of the classes of  $G/\text{Stab}(\alpha)$  if the set of the left cosets  $t\text{Stab}(\alpha)$  with  $t \in T$  is a partition of  $G$ , i.e.,

$$G = \bigsqcup_{t \in T} t\text{Stab}(\alpha).$$

**Definition 2.1.** Let  $T$  be a subset of  $G$  and  $\alpha_0 \in \Gamma$ . A pair  $(\alpha_0, T)$  is a *coordinate system* on  $\Gamma$  if  $T$  is a complete system of representatives of the classes of  $G/\text{Stab}(\alpha_0)$  and if  $1_G \in T$ , where  $1_G$  denotes the neutral element of  $G$ .

The element  $\alpha_0$  is called the *origin* of  $(\alpha_0, T)$  and the set  $T$  is called the *coordinate set* of  $(\alpha_0, T)$ . Since the action of  $G$  on  $\Gamma$  is transitive, for any  $\alpha \in \Gamma$ , there exists a unique  $t \in T$  such that  $t \cdot \alpha_0 = \alpha$  and  $t$  is called the *coordinate* of  $\alpha$  in the coordinate system  $(\alpha_0, T)$ .

**Example 2.2.** (a) For any group  $G$ , consider the action of  $G$  on itself by left multiplication. Then  $(1_G, G)$  is a coordinate system on  $G$ . This is the coordinate system used in the classical case. More generally, if  $\Gamma$  is a set equipped with a free left action of a group  $G$ , the pair  $(g_0, G)$  is a coordinate system on  $\Gamma$ , for any  $g_0 \in G$ .

(b) Denote by  $\text{Isom}(\mathbb{R}^d)$  the isometry group of  $\mathbb{R}^d$ . Let  $\Gamma = \mathbb{Z}^d$  and  $G \subset \text{Isom}(\mathbb{R}^d)$  be the subgroup of isometries preserving  $\Gamma$ . Define  $T \subset G$  as being the set of the translations in  $G$ . Then the pair  $(\alpha, T)$  is a coordinate system on  $\Gamma$ , for any  $\alpha \in \Gamma$ .

(c) Here is an example of a coordinate system  $(\alpha, T')$  where  $T'$  is not a subgroup of  $G$ . Let us take the previous example with  $d = 2$ , and denote by  $T_1 \subset T$  the subset of translations  $t \in T$  such that  $t \cdot (0, 0) \in \mathbb{N}^* \times \mathbb{N}$  and by  $r \in G$  the rotation about  $(0, 0)$  by the angle  $\frac{\pi}{2}$ . Then the pair  $((0, 0), T')$  is a coordinate system on  $\Gamma$ , with

$$T' = T_1 \cup rT_1 \cup r^2T_1 \cup r^3T_1 \cup \{\text{Id}_{\mathbb{Z}^2}\}.$$

(d) Denote by  $\mathbb{H}^d$  the  $d$ -dimensional hyperbolic space, by  $\text{Isom}(\mathbb{H}^d)$  the isometry group of  $\mathbb{H}^d$  and by  $\text{Isom}^+(\mathbb{H}^d)$  (resp.  $\text{Isom}^-(\mathbb{H}^d)$ ) the subset of isometries preserving (resp. reversing) the orientation. Note that  $\text{Isom}^+(\mathbb{H}^d)$  is a subgroup of  $\text{Isom}(\mathbb{H}^d)$ . A tessellation of  $\mathbb{H}^d$  is a tiling of  $\mathbb{H}^d$  by congruent polytopes such that the reflections with respect to the faces of the polytopes preserve the tiling. Let  $\Gamma$  be the set of polytopes of a tessellation of  $\mathbb{H}^d$  and  $G$  be the subgroup of  $\text{Isom}(\mathbb{H}^d)$

preserving the tessellation. Choose a polytope  $\alpha_0 \in \Gamma$  of the tessellation and let  $T$  be the Coxeter group generated by the reflections with respect to the faces of  $\alpha_0$ . Suppose the hypothesis of Poincaré's theorem are verified (see e.g. [3]). Then  $T$  is a normal subgroup of  $G$  and the pair  $(\alpha_0, T)$  is a coordinate system on  $\Gamma$ .

(e) In the previous example, suppose there exists a reflection  $r_0 \in \text{Stab}(\alpha_0)$  preserving the polytope  $\alpha_0$ . Denote by  $T^+ = T \cap \text{Isom}^+(\mathbb{H}^d)$  the subgroup of orientation-preserving isometries of  $T$  and by  $T^- = T \cap \text{Isom}^-(\mathbb{H}^d)$  the subset of orientation-reversing isometries of  $T$ . Define  $T' \subset G$  as  $T' = T^+ \cup (T^- r_0)$ . Then  $T'$  is a subgroup of  $\text{Isom}^+(\mathbb{H}^d)$  and the pair  $(\alpha_0, T')$  is a coordinate system on  $\Gamma$ .

*Remark 2.3.* If the pair  $(\alpha_0, T)$  is a coordinate system on  $\Gamma$  then, for any  $g \in T$ , the pair  $(g \cdot \alpha_0, Tg^{-1})$  is also a coordinate system on  $\Gamma$ .

*Remark 2.4.* If the pair  $(\alpha_0, T)$  is a coordinate system on  $\Gamma$  then, for any  $g \in G$ , the pair  $(g \cdot \alpha_0, gTg^{-1})$  is also a coordinate system on  $\Gamma$ . These remarks give a simple way to change the origin of a coordinate system, if needed.

Denote by  $\text{Stab}(\alpha_0, H)$  the stabilizer subgroup of  $\alpha_0$  in  $H$ , for a subgroup  $H$  of  $G$ . We have  $\text{Stab}(\alpha_0, H) = \text{Stab}(\alpha_0) \cap H$ . Remark that  $T \cap \text{Stab}(\alpha_0)$  is the trivial subgroup of  $G$  for any coordinate system  $(\alpha_0, T)$ .

We can decompose each element of  $G$  as a product of an element of  $T$  and an element of the stabilizer subgroup of  $\alpha_0$ , i.e., for any  $g \in G$ , there exist  $t \in T$  and  $r \in \text{Stab}(\alpha_0)$  such that  $g = tr$ . More generally, there is a similar decomposition for any subgroup of  $G$ .

**Proposition 2.5.** *Let  $H$  be a subgroup of  $G$ . For any coordinate system  $(\alpha_0, T)$  on  $\Gamma$  such that  $T \subset H$ , we have  $H = T \cdot \text{Stab}(\alpha_0, H)$ .*

*Proof.* Let  $h$  be an element in  $H$ . Since  $(\alpha_0, T)$  is a coordinate system on  $\Gamma$ , there exists a unique  $t \in T$  such that  $t \cdot \alpha_0 = h \cdot \alpha_0$ . Then  $\alpha_0 = t^{-1}h \cdot \alpha_0$  and consequently  $t^{-1}h \in \text{Stab}(\alpha_0)$ . We have  $t^{-1}h \in H$  because  $t \in T \subset H$ . Since  $h = t \cdot (t^{-1}h)$ , we have  $H = T \cdot \text{Stab}(\alpha_0, H)$ .  $\square$

Note that such a decomposition of  $h$  in  $T \cdot \text{Stab}(\alpha_0, H)$  is unique.

*Remark.* With the hypothesis given in Proposition 2.5, if  $T$  is a normal subgroup of  $H$ , then  $H$  is the semidirect product of  $T$  and  $\text{Stab}(\alpha_0)$ .

### 3. CELLULAR AUTOMATA

Let  $\Gamma$  be a set equipped with a transitive left action of a group  $G$ . For  $g \in G$ , let  $L_g: \Gamma \rightarrow \Gamma$  denote the map defined by  $L_g(\alpha) = g \cdot \alpha$  for all  $\alpha \in \Gamma$ .

Let  $Q$  be a nonempty finite set. Consider the set  $Q^\Gamma$  consisting of all maps from  $\Gamma$  to  $Q$ :

$$Q^\Gamma = \prod_{\alpha \in \Gamma} Q = \{x: \Gamma \rightarrow Q\}.$$

The elements of  $Q$  are called the *states*. The set  $\Gamma$  is the *universe* and its elements are called the *cells*. The elements of  $Q^\Gamma$  are called the *configurations*.

Given an element  $g \in G$  and a configuration  $x \in Q^\Gamma$ , we define the configuration  $gx \in Q^\Gamma$  by

$$gx = x \circ L_{g^{-1}}.$$

This defines a left group action of  $G$  on  $Q^\Gamma$ .

**Definition 3.1.** A *cellular automaton* over the state set  $Q$  and the universe  $\Gamma$  is a map  $\tau: Q^\Gamma \rightarrow Q^\Gamma$  satisfying the following property: there exists a coordinate system  $(\alpha_0, T)$ , a finite subset  $M \subset \Gamma$  and a map  $\mu: Q^M \rightarrow Q$  such that

$$\tau(x)(\alpha) = \mu((t^{-1}x)|_M) \quad (3.1)$$

for all  $x \in Q^\Gamma$  and  $\alpha \in \Gamma$ , where  $t \in T$  denotes the coordinate of  $\alpha$  and  $(t^{-1}x)|_M$  denotes the restriction of the configuration  $t^{-1}x$  to  $M$ .

Such a set  $M$  is called a *memory set* for  $\tau$ , and  $\mu$  is called a *local defining map* for  $\tau$ . For  $\alpha = \alpha_0$ , formula (3.1) gives us

$$\tau(x)(\alpha_0) = \mu(x|_M) \quad (3.2)$$

for all  $x \in Q^\Gamma$  since the coordinate of the origin  $\alpha_0$  is  $1_G$ . Thus, by formulas (3.1) and (3.2), we have

$$\tau(x)(\alpha) = \tau(t^{-1}x)(\alpha_0) \quad (3.3)$$

for all  $x \in Q^\Gamma$  and  $\alpha \in \Gamma$ , where  $t \in T$  denotes the coordinate of  $\alpha$ . Following the definition of the left action of  $G$  on  $Q^\Gamma$  above, one has  $\tau(x)(\alpha) = \tau(x)(t \cdot \alpha_0) = t^{-1}\tau(x)(\alpha_0)$ , and consequently,

$$\tau(t^{-1}x)(\alpha_0) = t^{-1}\tau(x)(\alpha_0) \quad (3.4)$$

for all  $x \in Q^\Gamma$  and  $t \in T$ .

*Remark 3.2.* Most cellular automata are constructed this way: given a finite subset  $M \subset \Gamma$ , a map  $\mu: Q^M \rightarrow Q$  and a coordinate system  $(\alpha_0, T)$ , one defines the map  $\tau: Q^\Gamma \rightarrow Q^\Gamma$  by setting

$$\tau(x)(\alpha) = \mu((t^{-1}x)|_M)$$

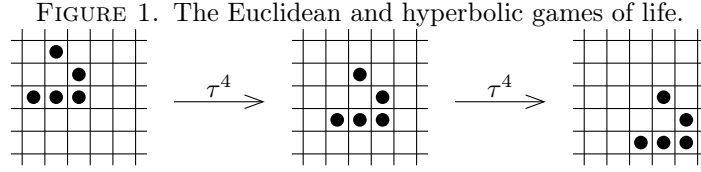
for all  $x \in Q^\Gamma$  and  $\alpha \in \Gamma$ , where  $t \in T$  denotes the coordinate of  $\alpha$ . The map  $\tau$  is clearly a cellular automaton. Such a triple  $(M, \mu, (\alpha_0, T))$  is called a *construction triple* for the cellular automaton  $\tau$ . Two construction triples are called equivalent if they give rise to the same cellular automaton. This defines an equivalence relation. There is a one-to-one correspondance between the equivalence classes of construction triples and the cellular automata on  $Q^\Gamma$ . Note that it is quite common to define a cellular automaton  $A$  as an equivalence class of construction triples  $A = [(M, \mu, (\alpha_0, T))]$ . Many papers use this definition without mentioning it, as it is supposed to be known, but you may still see [6]. In this case, the map  $\tau$  is called the *global transition map* of  $A$ .

**Example 3.3.** (a) **A hyperbolic Game of Life cellular automaton.** This one is adapted from the famous Conway's Game of Life cellular automaton, which was proved to be universal in [1]. Consider a tessellation of  $\mathbb{H}^2$  by regular octagons. Let  $\Gamma$  be the set of the polygons of the tessellation and  $G \subset \text{Isom}(\mathbb{H}^2)$  be the subgroup of isometries preserving  $\Gamma$ . Let  $(\alpha_0, T)$  be a coordinate system for  $\Gamma$  and define  $M$  as the set of polygons having a common edge with  $\alpha_0$  (this includes  $\alpha_0$  itself). Consider the state set  $Q = \{0, 1\}$ . For a configuration  $x \in Q^\Gamma$ , one says that a cell  $\alpha$  is alive if  $x(\alpha) = 1$  and dead otherwise. Consider the map  $\mu: Q^M \rightarrow Q$  defined as follow:

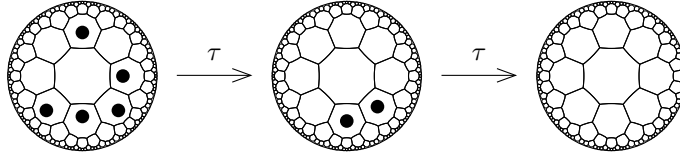
$$\mu(x) = \begin{cases} 1 & \text{if } \begin{cases} \sum_{\beta \in M} x(\beta) = 3 \\ \text{or} \\ \sum_{\beta \in M} x(\beta) = 4 \text{ and } x(\alpha_0) = 1 \end{cases} \\ 0 & \text{otherwise} \end{cases}$$

for all  $x \in Q^M$ . The construction triple  $(M, \mu, (\alpha_0, T))$  defines a cellular automaton over the state set  $Q$  and the universe  $\Gamma$ . This cellular automaton can be interpreted

as for its Euclidean version: the neighborhood of a cell consists of the cells having an edge in common with it; if a cell is alive in the configuration  $x$ , then the cell dies in the configuration  $\tau(x)$  if it is overcrowded (i.e., it has 4 or more neighbor cells alive) or lonely (i.e., it has 1 or 0 neighbor cell alive) in the configuration  $x$ ; it remains alive otherwise; if a cell is dead in the configuration  $x$ , then the cell is reborn in the configuration  $\tau(x)$  if it has 3 neighbor cells alive in the configuration  $x$ ; it remains dead otherwise. Remark that if the angles of the tessellation are  $\frac{2\pi}{p}$ , with  $p$  an even number, then the action of the Coxeter group is free and  $\tau$  is a cellular automaton in the classical definition.



*In the euclidian plan, the glider translates itself after 4 steps, and goes on infinitely.*



*When transferred in the hyperbolic plan, the glider just crashes and disappears after 2 steps.*

(b) **The Fairy Lights cellular automaton.** Consider  $\Gamma = \mathbb{Z}^2$  and  $G \subset \text{Isom}(\mathbb{R}^2)$  as defined in Example 2.2 (b). For  $\alpha \in \mathbb{Z}^2$ , denote by  $t_\alpha: \mathbb{Z}^2 \rightarrow \mathbb{Z}^2$  the translation defined by  $t_\alpha(\beta) = \beta + \alpha$  and let

$$T_1 = \{t_{(\alpha_1, \alpha_2)}: \alpha_1 + \alpha_2 \in 2\mathbb{Z}\}$$

and

$$T_2 = \{(-\text{Id}_{\mathbb{Z}^2}) \circ t_{(\alpha_1, \alpha_2)}: \alpha_1 + \alpha_2 \in 2\mathbb{Z} + 1\}.$$

Then the pair  $(\alpha_0, T)$  is a coordinate system on  $\Gamma$ , with  $\alpha_0 = (0, 0)$  and  $T = T_1 \cup T_2$ . The cells of  $\Gamma$  represent bulbs that are turned on. The set  $Q$  represents the possible colors of a bulb. Let  $M = \{(0, 1)\}$  and consider the map  $\mu: Q^M \rightarrow Q$  defined as follow:

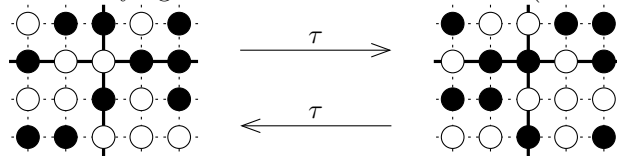
$$\mu(x) = x((0, 1))$$

for all  $x \in Q^M$ . The construction triple  $(M, \mu, (\alpha, T))$  defines a cellular automaton  $\tau$  over the state set  $Q$  and the universe  $\mathbb{Z}^2$  and we have

$$\tau(x)(\alpha_1, \alpha_2) = \begin{cases} x(\alpha_1, \alpha_2 + 1) & \text{if } \alpha_1 + \alpha_2 \in 2\mathbb{Z} \\ x(\alpha_1, \alpha_2 - 1) & \text{otherwise} \end{cases}.$$

Note that  $\tau \circ \tau = \text{Id}_{(\mathbb{Z}^2)_Q}$  and thus  $\tau$  is reversible (see Section 6).

FIGURE 2. The fairy lights automaton  $\tau$  with 2 colors (black and white).

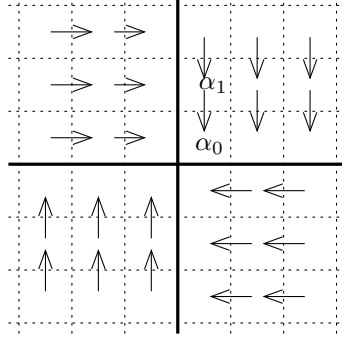


(c) **A state shift cellular automaton.** A state shift cellular automaton is a cellular automaton whose memory set  $M$  is a singleton and whose local defining map is the identification  $Q^M \simeq Q$ . Consider the tessellation of the Euclidean plane  $\mathbb{R}^2$  by unit squares with vertices in  $\mathbb{Z}^2$ . Let  $\Gamma$  be the set of the squares of the tessellation and  $G \subset \text{Isom}(\mathbb{R}^2)$  be the subgroup of isometries preserving  $\Gamma$ . Denote by  $t_a: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  the translation defined by  $t_a(b) = b + a$  for all  $a$  and  $b \in \mathbb{R}^2$  and let  $T_1 = \{t_a \in G: a \in \mathbb{N}^2\}$  and  $r \in G$  be the rotation about  $(0,0)$  by the angle  $\frac{\pi}{2}$ . Then the pair  $(\alpha_0, T)$  is a coordinate system on  $\Gamma$ , with  $\alpha_0$  the square of  $\Gamma$  whose center is  $(\frac{1}{2}, \frac{1}{2})$  and  $T = T_1 \cup rT_1 \cup r^2T_1 \cup r^3T_1$ . Let  $Q$  be a nonempty finite set and  $M = \{\alpha_1\}$ , with  $\alpha_1$  the square of  $\Gamma$  whose center is  $(\frac{1}{2}, \frac{3}{2})$ . Consider the map  $\mu: Q^M \rightarrow Q$  defined as follow:

$$\mu(x) = x(\alpha_1)$$

for all  $x \in Q^M$ . The construction triple  $(M, \mu, (\alpha_0, T))$  defines a cellular automaton over the state set  $Q$  and the universe  $\Gamma$ . This automaton shifts the state of a cell of the first quadrant to the cell below, the state of a cell of the second quadrant to the cell on its right, the state of a cell of the third quadrant to the cell above, and the state of a cell of the fourth quadrant to the cell on its left (see figure 3).

FIGURE 3. The state shift automaton of Example (c)



The arrows symbolize the displacement of the states by the action of  $\tau$ .

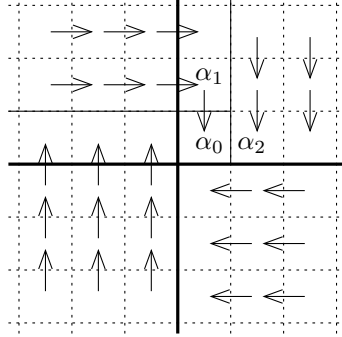
(d) **Another state shift cellular automaton.** Consider the same tessellation  $\Gamma$  of the Euclidean plane  $\mathbb{R}^2$  by unit squares and vertices in  $\mathbb{Z}^2$  and the same subgroup  $G \subset \text{Isom}(\mathbb{R}^2)$ . Let  $T_1 = \{t_a \in G: a \in \mathbb{N}^* \times \mathbb{N}\}$  (where  $t_a$  still denotes the same translation) and  $r \in G$  be the rotation about  $(\frac{1}{2}, \frac{1}{2})$  by the angle  $\frac{\pi}{2}$ . Then the pair  $(\alpha_0, T)$  is a coordinate system on  $\Gamma$ , with  $\alpha_0$  the square of  $\Gamma$  whose center is  $(\frac{1}{2}, \frac{1}{2})$  and  $T = T_1 \cup rT_1 \cup r^2T_1 \cup r^3T_1 \cup \{\text{Id}_{\mathbb{R}}\}$ . Let  $Q$  be a nonempty finite set and  $M = \{\alpha_1\}$ , with  $\alpha_1$  the square of  $\Gamma$  whose center is  $(\frac{1}{2}, \frac{3}{2})$ . Consider the map  $\mu: Q^M \rightarrow Q$  defined as follow:

$$\mu(x) = x(\alpha_1)$$

for all  $x \in Q^M$ . The construction triple  $(M, \mu, (\alpha_0, T))$  defines a cellular automaton  $\tau$  over the state set  $Q$  and the universe  $\Gamma$ . This automaton is very similar to the previous one (see figure 4), but differs on this:  $\tau \circ \tau$  is not a cellular automaton (see Section 7).

(e) **The Margolus billiard-ball cellular automaton.** We still consider the same tessellation  $\Gamma$  of the Euclidean plane  $\mathbb{R}^2$  by unit squares and  $G \subset \text{Isom}(\mathbb{R}^2)$  the subgroup of isometries preserving  $\Gamma$ . Denote by  $r \in G$  the rotation about

FIGURE 4. The state shift automaton of Example (d)



The arrows symbolize the displacement of the states by the action of  $\tau$ .

$(1, 1)$  by the angle  $\frac{\pi}{2}$ . Let  $\alpha_0$  be the square of  $\Gamma$  whose center is  $(\frac{1}{2}, \frac{1}{2})$  and  $T = \{t_a \in G : a \in 2\mathbb{Z} \times 2\mathbb{Z}\}$  (where  $t_a$  still denotes the same translation). Then the pair  $(\alpha_0, T_0)$  is a coordinate system on  $\Gamma$ , with  $T_0 = T \cup Tr \cup Tr^2 \cup Tr^3$ . Let  $Q = \{0, 1\}$  and  $M_0 = \{\alpha_0, r \cdot \alpha_0, r^2 \cdot \alpha_0, r^3 \cdot \alpha_0\}$ . Consider the map  $\mu_0 : Q^{M_0} \rightarrow Q$  defined as follow:

$$\mu_0(x) = \begin{cases} x(r^2 \cdot \alpha_0) & \text{if } \sum_{\alpha \in M_0} x(\alpha) = 1 \\ x(r \cdot \alpha_0) & \text{if } \sum_{\alpha \in M_0} x(\alpha) = 2 \text{ and } x(r \cdot \alpha_0) = x(r^3 \cdot \alpha_0) \\ x(\alpha_0) & \text{otherwise} \end{cases}$$

for all  $x \in Q^{M_0}$ . The construction triple  $(M_0, \mu_0, (\alpha_0, T_0))$  defines a cellular automaton  $\tau_0$  over the state set  $Q$  and the universe  $\Gamma$ . Note that  $\tau_0$  is involutive since  $\tau_0 \circ \tau_0 = \text{Id}_{Q^\Gamma}$  and therefore  $\tau_0$  is a reversible cellular automaton (see Section 6). Let  $t_0 \in G$  be the translation  $t_{(1,1)}$  and define the map  $\tau_1 : Q^\Gamma \rightarrow Q^\Gamma$  by  $\tau_1(x) = t_0 \tau_0(t_0^{-1}x)$  for all  $x \in Q^\Gamma$ . The map  $\tau_1$  is a cellular automaton since  $(t_0 \cdot M_0, t_0 \mu_0, (t_0 \cdot \alpha_0, t_0 T_0 t_0^{-1}))$  is a construction triple for  $\tau_1$ , where  $t_0 \mu_0 : Q^{t_0 \cdot M_0} \rightarrow Q$  is defined by  $t_0 \mu_0(x) = \mu_0(t_0^{-1}x)$  for all  $x \in Q^{t_0 \cdot M_0}$ . As  $\tau_0$  is involutive, we also have  $\tau_1 \circ \tau_1 = \text{Id}_{Q^\Gamma}$ . The Margolus billiard-ball cellular automaton is the map  $\tau = \tau_1 \circ \tau_0$  (see figure 5). It will be proved in Section 7 that  $\tau$  is a cellular automaton.

(f) Let  $\Gamma$  be a set equipped with a transitive left action of a group  $G$  and  $Q$  be any finite set. Let  $(\alpha_0, T)$  be any coordinate system on  $\Gamma$ . With  $M = \{\alpha_0\}$  and  $\mu : Q^M \rightarrow Q$  defined by

$$\mu(x) = x(\alpha_0)$$

for all  $x \in Q^M$ . Then the cellular automaton defined by the construction triple  $(M, \mu, (\alpha, T))$  is the identity map  $\tau = \text{Id}_{Q^\Gamma}$ .

Given a map  $\tau : Q^\Gamma \rightarrow Q^\Gamma$ , we will denote by  $Eq(\tau)$  the subset of  $G$  defined by

$$Eq(\tau) = \{g \in G : \tau(gx) = g\tau(x) \text{ for all } x \in Q^\Gamma\}.$$

**Proposition 3.4.** *For any map  $\tau : Q^\Gamma \rightarrow Q^\Gamma$ , the set  $Eq(\tau)$  is a subgroup of  $G$ .*

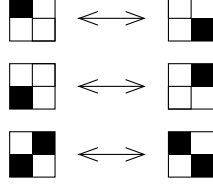
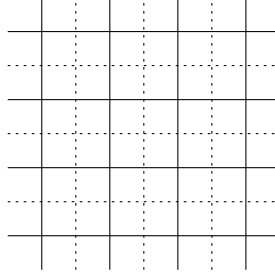
*Proof.* It is clear that  $1_G \in Eq(\tau)$ . Given  $g_1$  and  $g_2 \in Eq(\tau)$ , we have

$$\tau(g_1 g_2 x) = g_1 \tau(g_2 x) = g_1 g_2 \tau(x)$$

for any  $x \in Q^\Gamma$ , consequently  $g_1 g_2 \in Eq(\tau)$ . Finally, if  $g \in Eq(\tau)$ , one has

$$g\tau(g^{-1}x) = \tau(gg^{-1}x) = \tau(x)$$

FIGURE 5. The Margolus billiard-ball automaton

The Margolus billiard-ball rules for  $\tau_1$  and  $\tau_2$ .The Margolus billiard-ball rules are applied in the plain grid for  $\tau_1$  and the dash grid for  $\tau_2$ .

and then  $\tau(g^{-1}x) = g^{-1}\tau(x)$ . Therefore  $g^{-1} \in Eq(\tau)$  and  $Eq(\tau)$  is a subgroup of  $G$ .  $\square$

**Definition 3.5.** Let  $H$  be a subgroup of the group  $G$ . One says that a map  $\tau: Q^\Gamma \rightarrow Q^\Gamma$  is  $H$ -equivariant if  $H \subset Eq(\tau)$ , i.e., for all  $h \in H$  and for all  $x \in Q^\Gamma$ , we have  $\tau(hx) = h\tau(x)$ .

This can also be written  $\tau(x \circ L_{h^{-1}}) = \tau(x) \circ L_{h^{-1}}$ , or  $\tau(hx)(\alpha) = \tau(x)(h^{-1} \cdot \alpha)$  for all  $\alpha \in \Gamma$ .

We can characterize the  $H$ -equivariance of a cellular automaton by the  $H$ -invariance of any of its local defining map, defined as follows.

**Definition 3.6.** Let  $S$  be a subset of  $G$  and let  $\Omega$  be a subset of  $\Gamma$ . One says that a map  $\varphi: Q^\Omega \rightarrow Q$  is  $S$ -invariant if for all  $s \in S$  and for all  $x \in Q^\Omega$ , we have  $\varphi(sx|_\Omega) = \varphi(x|_\Omega)$ .

**Proposition 3.7.** Let  $S$  be a subset of  $G$  and let  $\Omega$  be a subset of  $\Gamma$ . Denote by  $H$  the subgroup of  $G$  generated by  $S$ . Let  $\varphi: Q^\Omega \rightarrow Q$  be a map. Then the following conditions are equivalent:

- (i) the map  $\varphi$  is  $H$ -equivariant;
- (ii) the map  $\varphi$  is  $S$ -invariant.

*Proof.* (i) $\Rightarrow$ (ii) is obvious. Conversely, suppose (ii), i.e.,  $\varphi$  is  $S$ -invariant. As any element of  $H$  is  $s_1^{a_1}s_2^{a_2}\dots s_p^{a_p}$  with  $a_i \in \mathbb{Z}$  and  $s_i \in S$ , it is sufficient to prove that  $\varphi$  is  $S^{-1}$ -invariant. For any  $s \in S$ , one has

$$\varphi(x|_\Omega) = \varphi(ss^{-1}x|_\Omega) = \varphi(s^{-1}x|_\Omega)$$

and therefore  $\varphi$  is  $S^{-1}$ -invariant.  $\square$

**Proposition 3.8.** Let  $\tau: Q^\Gamma \rightarrow Q^\Gamma$  be a cellular automaton and  $(M, \mu, (\alpha_0, T))$  be a construction triple for  $\tau$ . Suppose that  $H$  is a subgroup of  $G$  containing  $T$ . Then the following conditions are equivalent:

- (i) the map  $\tau$  is  $H$ -equivariant;
- (ii) the map  $\mu$  is  $\text{Stab}(\alpha_0, H)$ -invariant.



*Proof.* Suppose first that the map  $\mu$  is  $\text{Stab}(\alpha_0, H)$ -invariant. Let  $h \in H$ ,  $x \in Q^\Gamma$ , and  $\alpha \in \Gamma$ . Let  $t \in T \subset H$  be the coordinate of  $\alpha$ . By Proposition 2.5, one has  $H = T \cdot \text{Stab}(\alpha_0, H)$ . As  $h^{-1}t \in H$ , we can write  $h^{-1}t = t's$  for some  $t' \in T$  and  $s \in \text{Stab}(\alpha_0, H)$ . Consequently  $h^{-1} \cdot \alpha = h^{-1}t \cdot \alpha_0 = t's \cdot \alpha_0 = t' \cdot \alpha_0$  and

$$h\tau(x)(\alpha) = \tau(x)(h^{-1} \cdot \alpha) = \tau(x)(t' \cdot \alpha_0) = \mu((t'^{-1}x)|_M).$$

On the other hand, since  $\mu$  is  $\text{Stab}(\alpha_0, H)$ -invariant, we have

$$\tau(hx)(\alpha) = \mu((t^{-1}hx)|_M) = \mu((s^{-1}t'^{-1}x)|_M) = \mu((t'^{-1}x)|_M).$$

Hence  $h\tau(x)(\alpha) = \tau(hx)(\alpha)$  for all  $\alpha \in \Gamma$ . Thus  $h\tau(x) = \tau(hx)$  for all  $h \in H$  and for all  $x \in Q^\Gamma$  and therefore  $\tau$  is  $H$ -equivariant.

Conversely, suppose that  $\tau$  is  $H$ -equivariant, i.e., for all  $h \in H$ , for all  $x \in Q^\Gamma$ , and for all  $\alpha \in \Gamma$ , we have  $\tau(hx)(\alpha) = \tau(x)(h^{-1} \cdot \alpha)$ . Let  $x \in Q^\Gamma$  and  $s \in \text{Stab}(\alpha_0, H)$ . We have  $\mu(x|_M) = \tau(x)(\alpha_0)$  and

$$\mu(sx|_M) = \tau(sx)(\alpha_0) = \tau(x)(s^{-1} \cdot \alpha_0) = \tau(x)(\alpha_0).$$

Thus  $\mu(sx|_M) = \mu(x|_M)$  and  $\mu$  is  $\text{Stab}(\alpha_0, H)$ -invariant.  $\square$

#### 4. EQUIVARIANT CELLULAR AUTOMATON

Let  $\Gamma$  be a set equipped with a transitive left action of a group  $G$  and let  $Q$  be a nonempty finite set.

**Definition 4.1.** Let  $\tau: Q^\Gamma \rightarrow Q^\Gamma$  be a cellular automaton and  $(\alpha_0, T)$  be a coordinate system on  $\Gamma$ . One says that  $(\alpha_0, T)$  is a coordinate system for  $\tau$  if there exists a finite subset  $M \subset \Gamma$  and a map  $\mu: Q^\Gamma \rightarrow Q$  such that  $(M, \mu, (\alpha_0, T))$  is a construction triple for  $\tau$ .

**Proposition 4.2.** Let  $\tau: Q^\Gamma \rightarrow Q^\Gamma$  be a cellular automaton. Then for any cell  $\alpha_0 \in \Gamma$ , there exists a subset  $T \subset G$  such that the pair  $(\alpha_0, T)$  is a coordinate system for  $\tau$ .

*Proof.* Let  $(M, \mu, (\alpha_1, U))$  be a construction triple for  $\tau$ . As the pair  $(\alpha_1, U)$  is a coordinate system on  $\Gamma$ , there exists  $g \in U$  such that  $g \cdot \alpha_1 = \alpha_0$ . From Remark 2.3, the pair  $(\alpha_0, T)$  is a coordinate system on  $\Gamma$ , with  $T = Ug^{-1}$ . Then  $(\alpha_0, T)$  is a coordinate system for  $\tau$ . Indeed, let's define the map  $\tilde{\mu}: Q^{g \cdot M} \rightarrow Q$  as follow:  $\tilde{\mu}(x) = \mu(g^{-1}x)$  for all  $x \in Q^{g \cdot M}$ . Then from (3.1) we have

$$\tau(x)(\alpha) = \mu((u^{-1}x)|_M) = \tilde{\mu}((gu^{-1}x)|_{g \cdot M}) = \tilde{\mu}((t^{-1}x)|_{g \cdot M})$$

for all  $x \in Q^\Gamma$  and for all  $\alpha \in \Gamma$ , where  $u$  denotes the coordinate of  $\alpha$  in  $(\alpha_1, U)$  and  $t = ug^{-1}$  denotes the coordinate of  $\alpha$  in  $(\alpha_0, T)$ . Thus  $(g \cdot M, \tilde{\mu}, (\alpha_0, T))$  is a construction triple for  $\tau$ .  $\square$

Proposition 4.2 shows that one can choose the origin of a coordinate system for a cellular automaton. This property will be used throughout this paper. The following proposition shows that the memory set and the local defining map only depend on the origin of the coordinate system:

**Proposition 4.3.** Let  $\tau: Q^\Gamma \rightarrow Q^\Gamma$  be a cellular automaton and  $(M, \mu, (\alpha_0, T))$  be a construction triple for  $\tau$ . Let  $(\alpha_0, U)$  be another coordinate system on  $\Gamma$ . Then the following hold:

- (i) if  $(\alpha_0, U)$  is a coordinate system for  $\tau$ , then  $(M, \mu, (\alpha_0, U))$  is another construction triple for  $\tau$ ;
- (ii) if  $U \subset \text{Eq}(\tau)$ , then  $(\alpha_0, U)$  is a coordinate system for  $\tau$ .

*Proof.* Suppose first that  $(\alpha_0, U)$  is a coordinate system for  $\tau$ . Let  $x \in Q^\Gamma$  be a configuration and  $\alpha \in \Gamma$  be a cell with coordinate  $u \in U$  in the coordinate system  $(\alpha_0, U)$ . By formula (3.3) we have

$$\begin{aligned}\tau(x)(\alpha) &= \tau(u^{-1}x)(\alpha_0) \\ &= \mu(u^{-1}x|_M)\end{aligned}$$

and thus  $(M, \mu, (\alpha_0, U))$  is another construction triple for  $\tau$ .

Suppose now that  $U \subset Eq(\tau)$ . Let  $x \in Q^\Gamma$  be a configuration and  $\alpha \in \Gamma$  be a cell with coordinate  $u \in U$  in the coordinate system  $(\alpha_0, U)$ . Then we have

$$\begin{aligned}\tau(x)(\alpha) &= \tau(x)(u \cdot \alpha_0) \\ &= u^{-1}\tau(x)(\alpha_0) \\ &= \tau(u^{-1}x)(\alpha_0)\end{aligned}$$

since  $u \in Eq(\tau)$ . By formula (3.2) one has

$$\tau(x)(\alpha) = \mu(u^{-1}x|_M)$$

and thus  $(M, \mu, (\alpha_0, U))$  is a construction triple for  $\tau$  and the pair  $(\alpha_0, U)$  is a coordinate system for  $\tau$ .  $\square$

As the restriction map  $Q^\Gamma \rightarrow Q^M$ ,  $x \mapsto x|_M$  is surjective, formula (3.2) shows that if  $M$  is a memory set for a cellular automaton  $\tau$  and  $\alpha_0 \in \Gamma$  is a cell, then there is a unique map  $\mu: Q^M \rightarrow Q$  which satisfies (3.1). Thus one says that  $\mu$  is the local defining map for  $\tau$  associated with the memory set  $M$  and the origin  $\alpha_0$ .

Proposition 4.3 shows that if the subgroup  $Eq(\tau)$  contains a coordinate set, then the corresponding coordinate system on  $\Gamma$  is a coordinate system for  $\tau$ . In this case, from Proposition 4.2, we deduce that the subgroup  $Eq(\tau)$  contains many coordinate systems on  $\Gamma$ , at least one for each origin. A subgroup having this property will be qualified as “big”.

**Definition 4.4.** A subgroup  $H \subset G$  is called a *big subgroup* of  $G$  if the action of  $H$  on  $\Gamma$  induced by the action of  $G$  on  $\Gamma$  is transitive.

As a consequence of  $H$  being a big subgroup, for any origin  $\alpha_0 \in \Gamma$ , there exists a coordinate system  $(\alpha_0, T)$  on  $\Gamma$  such that  $H$  contains  $T$ .

**Definition 4.5.** One says that a cellular automaton  $\tau: Q^\Gamma \rightarrow Q^\Gamma$  is *equivariant* if  $Eq(\tau)$  is a big subgroup of  $G$ .

An equivariant cellular automaton has the property to be  $H$ -equivariant for some big subgroup  $H$  of  $G$ . For any coordinate system, denote by  $S(\alpha_0, T) = (T^{-1}T^{-1}T) \cap \text{Stab}(\alpha_0)$ . Then for all  $t, t' \in T$ , the coordinate of  $t^{-1}t' \cdot \alpha_0$  is  $t^{-1}t's^{-1}$  for some  $s \in S(\alpha_0, T)$ . We can characterize the equivariance of a cellular automaton by the  $S(\alpha_0, T)$ -invariance of its local defining map.

**Proposition 4.6.** Let  $\tau: Q^\Gamma \rightarrow Q^\Gamma$  be a cellular automaton. Then  $\tau$  is equivariant if and only if there exists a construction triple  $(M, \mu, (\alpha_0, T))$  for  $\tau$  such that the map  $\mu$  is  $S(\alpha_0, T)$ -invariant.

*Proof.* Suppose first that  $\tau$  is equivariant. Then by Proposition 4.3, there exists a construction triple  $(M, \mu, (\alpha_0, T))$  for  $\tau$  such that  $T \subset Eq(\tau)$ . Let  $s \in S(\alpha_0, T)$  and  $y \in Q^\Gamma$ . There exists  $t, t' \in T$  such that  $t^{-1}t's^{-1} \in T$ . Let  $x = t^{-1}t'y$ . One has

$$sy = s(t')^{-1}tx = (t^{-1}t's^{-1})^{-1}x$$

and then

$$\begin{aligned}\mu(sy|_M) &= \mu\left((t^{-1}t's^{-1})^{-1}x|_M\right) \\ &= \tau(x)(t^{-1}t's^{-1} \cdot \alpha_0) \\ &= (t')^{-1}t\tau(x)(s^{-1} \cdot \alpha_0).\end{aligned}$$

As  $T \subset Eq(\tau)$  and  $s \in \text{Stab}(\alpha_0)$ , we have

$$\begin{aligned}&= \tau\left((t')^{-1}tx\right)(\alpha_0) \\ &= \tau(y)(\alpha_0) \\ &= \mu(y|_M)\end{aligned}$$

and thus  $\mu$  is  $S(\alpha_0, T)$ -invariant.

Conversely, suppose now that there exists a construction triple  $(M, \mu, (\alpha_0, T))$  for  $\tau$  such that the map  $\mu$  is  $S(\alpha_0, T)$ -invariant. Let  $u \in T$  and  $x \in Q^\Gamma$ . For all  $t \in T$ , there exists  $s \in S(\alpha_0, T)$  such that  $u^{-1}ts^{-1} \in T$ . Then one has

$$\begin{aligned}u\tau(x)(t \cdot \alpha_0) &= \tau(x)(u^{-1}t \cdot \alpha_0) \\ &= \mu\left((u^{-1}ts^{-1})^{-1}x|_M\right) \\ &= \mu(st^{-1}ux|_M).\end{aligned}$$

As  $\mu$  is  $S(\alpha_0, T)$ -invariant, we have

$$\begin{aligned}&= \mu(t^{-1}ux|_M) \\ &= \tau(ux)(t \cdot \alpha_0)\end{aligned}$$

and thus  $u\tau(x) = \tau(ux)$  for all  $x \in Q^\Gamma$  and all  $u \in T$ . Therefore  $T \subset Eq(\tau)$  and  $\tau$  is equivariant.  $\square$

This proposition shall be used to prove that certain cellular automata are not equivariant, as shown in the following example.

**Example 4.7. Another state shift automaton.** Consider the tessellation of the Euclidean plane  $\mathbb{R}^2$  by unit squares and vertices in  $\mathbb{Z}^2$ . Let  $\Gamma$  be the set of the squares of the tessellation and  $G \subset \text{Isom}^+(\mathbb{R}^2)$  be the subgroup of direct isometries preserving  $\Gamma$ . Denote by  $t_a: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  the translation defined by  $t_a(b) = b + a$  for all  $a$  and  $b \in \mathbb{R}^2$  and let  $T_1 = \{t_{(a,b)} \in G: a, b \in \mathbb{Z} \text{ and } 0 \leq a \leq |b|\}$ . Also let  $r \in G$  be the rotation about  $(0,0)$  by the angle  $\frac{\pi}{2}$ . Then the pair  $(\alpha_0, T)$  is a coordinate system on  $\Gamma$ , with  $\alpha_0$  the square of  $\Gamma$  whose center is  $(\frac{1}{2}, \frac{1}{2})$  and  $T = T_1 \cup rT_1 \cup r^2T_1 \cup r^3T_1$ . Let  $Q$  be a nonempty finite set and  $M = \{\alpha_1\}$ , with  $\alpha_1$  the square of  $\Gamma$  whose center is  $(\frac{1}{2}, \frac{3}{2})$ . Consider the map  $\mu: Q^M \rightarrow Q$  defined as follow:

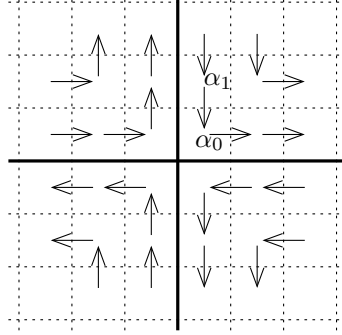
$$\mu(x) = x(\alpha_1)$$

for all  $x \in Q^M$ . The construction triple  $(M, \mu, (\alpha_0, T))$  defines a cellular automaton over the state set  $Q$  and the universe  $\Gamma$ . Note that a state shift automaton admits only one coordinate system in  $G$ . Since

$$r' = t_{(0,0)}^{-1} \circ s_{(1,0)} \circ r \in T^{-1}T^{-1}T$$

is the rotation about  $(\frac{1}{2}, \frac{1}{2})$  by the angle  $\frac{\pi}{2}$ , one has  $r' \in \text{Stab}(\alpha_0)$  and thus  $r' \in S(\alpha_0, T)$ . Therefore  $\mu$  is not  $S(\alpha_0, T)$ -invariant. Hence  $\tau$  is not equivariant by Proposition 4.6.

FIGURE 6. The state shift automaton of Example 4.7.



The arrows symbolize the displacement of the states by the action of  $\tau$ .

## 5. MINIMAL MEMORY SET

Let  $\Gamma$  be a set equipped with a transitive left action of a group  $G$  and  $Q$  a nonempty finite set.

From the definition of a memory set  $M$  for a cellular automaton  $\tau$  (cf. Definition 3.1), it is clear that if a subset  $M' \subset \Gamma$  contains  $M$ , then  $M'$  is also a memory set for  $\tau$ . It may happen that a subset  $M'' \subset M$  is also a memory set for  $\tau$ . We therefore define what is a “useful” element for the local defining map.

**Definition 5.1.** Let  $M$  be a subset of  $\Gamma$  and  $\mu: Q^M \rightarrow Q$  a map. A cell  $\alpha \in \Gamma$  is said to be  $\mu$ -useless if for all configurations  $x, y \in Q^\Gamma$  such that  $x|_{\Gamma \setminus \{\alpha\}} = y|_{\Gamma \setminus \{\alpha\}}$ , we have  $\mu(x|_M) = \mu(y|_M)$ . Otherwise,  $\alpha$  is said to be  $\mu$ -useful.

It is clear that any cell outside  $M$  is  $\mu$ -useless. Let  $\tau: Q^\Gamma \rightarrow Q^\Gamma$  be a cellular automaton and  $(M, \mu, (\alpha_0, T))$  be a construction triple for  $\tau$ . Denote by  $M_0$  the subset of  $M$  containing all the  $\mu$ -useful cells. Then  $M_0$  is also a memory set for  $\tau$  and is the minimal memory set of  $\tau$  for any coordinate system  $(\alpha_0, T')$  with respect to inclusion. More precisely, we have the following proposition:

**Proposition 5.2.** Let  $\tau: Q^\Gamma \rightarrow Q^\Gamma$  be a cellular automaton and  $(M, \mu, (\alpha_0, T))$  be a construction triple for  $\tau$ . Let  $M_0$  be the subset of  $M$  containing all the  $\mu$ -useful cells. Suppose  $(M', \mu', (\alpha_0, T'))$  is another construction triple for  $\tau$ . Then one has  $M_0 \subset M'$  and  $M_0$  is called the minimal memory set of  $\tau$  associated with the origin  $\alpha_0$ .

*Proof.* Suppose  $M_0 \not\subset M'$ . Let  $\beta \in M_0 \setminus M'$ . Since  $\beta$  is a  $\mu$ -useful cell, we may find two configurations  $x$  and  $y \in Q^\Gamma$  such that  $x|_{\Gamma \setminus \{\beta\}} = y|_{\Gamma \setminus \{\beta\}}$  and  $\mu(x|_M) \neq \mu(y|_M)$ . As  $\beta \notin M'$ , we have  $x|_{M'} = y|_{M'}$  and therefore  $\mu'(x|_{M'}) = \mu'(y|_{M'})$ . Hence  $\tau(x)(\alpha_0) = \mu'(x|_{M'}) = \mu'(y|_{M'}) = \tau(y)(\alpha_0)$  and then  $\mu(x|_M) = \tau(x)(\alpha_0) = \tau(y)(\alpha_0) = \mu(y|_M)$ , which contradicts the fact that  $\beta$  is  $\mu$ -useful.  $\square$

Note that originally, in the classical case, the memory set was defined as a neighborhood of the cell  $0_G$ , i.e., the nearest cells surrounding the cell  $0_G$ . Neighborhoods commonly used, when  $G = \mathbb{Z}^d$ , are the von Neumann neighborhood and the Moore neighborhood. The von Neumann neighborhood is defined with the  $\|\cdot\|_1$  metric and

the Moore neighborhood is defined with the  $\|\cdot\|_\infty$  metric, where  $\|x\|_1 = \sum_{k=1}^d |x_k|$  and

$\|x\|_\infty = \max_{k=1 \dots d} |x_k|$  for all  $x = (x_1, x_2, \dots, x_d) \in \mathbb{Z}^d$ . With this definition, the minimal memory set is not the set of the  $\mu$ -useful cells, but the smallest neighborhood containing the  $\mu$ -useful cells.

Proposition 5.2 shows that there is a unique minimal memory set for a given origin  $\alpha_0$ . For another origin  $\alpha_1$ , the minimal memory set is just a translation of the minimal memory set associated with  $\alpha_0$ .

**Proposition 5.3.** *Let  $\tau: Q^\Gamma \rightarrow Q^\Gamma$  be a cellular automaton. Assume that  $M$  and  $M'$  are minimal memory sets for  $\tau$ . Then one has  $M' = g \cdot M$  for some  $g \in G$ .*

*Proof.* Let  $(M, \mu, (\alpha_0, T))$  and  $(M', \mu', (\alpha_1, T'))$  be construction triples for  $\tau$ . By Proposition 4.2,  $(g \cdot M, \tilde{\mu}, (g \cdot \alpha_0, Tg^{-1}))$  is also a construction triple for  $\tau$  for all  $g \in T$ , where  $\tilde{\mu}$  is defined as in the proof of Proposition 4.2. Similarly,  $(g' \cdot M', \mu', (g' \cdot \alpha_1, T'g'^{-1}))$  is also a construction triple for  $\tau$  for all  $g' \in T'$ . Let  $g \in T$  denote the coordinate of  $\alpha_1$  in the coordinate system  $(\alpha_0, T)$  and  $g' \in T'$  denote the coordinate of  $\alpha_0$  in the coordinate system  $(\alpha_1, T')$ . Since  $M$  and  $M'$  are minimal memory sets, one has  $M \subset g' \cdot M'$  and  $M' \subset g \cdot M$ , and therefore  $M' \subset gg' \cdot M'$ . As  $|M'| = |gg' \cdot M'|$ , we have  $M' = gg' \cdot M'$  and then  $g \cdot M \subset gg' \cdot M' = M'$ . Thus  $M' = g \cdot M$ .  $\square$

Consequently, all the minimal memory sets have the same cardinality. The minimal memory set associated with the origin  $\alpha_0$  of an equivariant cellular automaton has the property of being  $S$ -invariant, with  $S$  the stabilizer subgroup of the origin  $\alpha_0$  in a big subgroup.

**Proposition 5.4.** *Let  $\tau: Q^\Gamma \rightarrow Q^\Gamma$  be an equivariant cellular automaton, and  $\alpha_0 \in \Gamma$ . Denote by  $M_0$  the minimal memory set associated with the origin  $\alpha_0$ . Let  $H \subset \text{Eq}(\tau)$  be a big subgroup of  $G$  and denote by  $S = \text{Stab}(\alpha_0, H)$ . Then  $M_0$  is  $S$ -invariant, i.e., one has  $S \cdot M_0 = M_0$ .*

*Proof.* Denote by  $\mu$  the local defining map associated with the memory set  $M_0$ . Let  $s \in S$  and  $\beta \in M_0$  and let's prove that the cell  $s \cdot \beta \in M_0$ , i.e.,  $s \cdot \beta$  is  $\mu$ -useful. As  $\beta$  is a  $\mu$ -useful cell, we may find two configurations  $x$  and  $y \in Q^\Gamma$  such that  $x|_{\Gamma \setminus \{\beta\}} = y|_{\Gamma \setminus \{\beta\}}$  and  $\mu(x|_{M_0}) \neq \mu(y|_{M_0})$ . Then  $sx|_{\Gamma \setminus \{s \cdot \beta\}} = sy|_{\Gamma \setminus \{s \cdot \beta\}}$  and since  $\mu$  is  $S$ -invariant by Proposition 3.8, we have  $\mu(sx|_{M_0}) = \mu(x|_{M_0}) \neq \mu(y|_{M_0}) = \mu(sy|_{M_0})$ . Therefore  $s \cdot \beta$  is  $\mu$ -useful.  $\square$

## 6. HEDLUND'S THEOREM

Let  $\Gamma$  be a set equipped with a transitive left action of a group  $G$  and  $Q$  a nonempty finite set.

We equip  $Q^\Gamma$  with the prodiscrete topology (i.e., the product topology where each factor  $Q$  of  $Q^\Gamma$  has the discrete topology). This is the smallest topology on  $Q^\Gamma$  for which the projection maps  $\pi_\alpha: Q^\Gamma \rightarrow Q$ , given by  $\pi_\alpha(x) = x(\alpha)$ , are continuous for every  $\alpha \in \Gamma$ . The elementary cylinders

$$\text{Cyl}(\alpha, q) = \{x \in Q^\Gamma : x(\alpha) = q\}$$

where  $\alpha \in \Gamma$  and  $q \in Q$  are both open and closed in  $Q^\Gamma$ . If  $x \in Q^\Gamma$ , a neighborhood base of  $x$  is given by the sets

$$V(x, \Omega) = \{y \in Q^\Gamma : x|_\Omega = y|_\Omega\} = \bigcap_{\alpha \in \Omega} \text{Cyl}(\alpha, x(\alpha))$$

where  $\Omega$  runs over all finite subsets of  $\Gamma$ .

An important feature of cellular automata is their continuity, with respect to the prodiscrete topology. We will use the following lemma in the proof of this property.

**Lemma 6.1.** *Let  $\tau: Q^\Gamma \rightarrow Q^\Gamma$  be a cellular automaton with memory set  $M$  and coordinate system  $(\alpha_0, T)$  and let  $\alpha \in \Gamma$ . Then  $\tau(x)(\alpha)$  only depends on the restriction of  $x$  to  $t \cdot M$ , where  $t \in T$  denotes the coordinate of  $\alpha$ .*

*Proof.* Since  $\tau(x)(\alpha) = \mu((t^{-1}x)|_M)$  and for all  $\beta \in M$ ,  $(t^{-1}x)(\beta) = (x \circ L_t)(\beta) = x(t \cdot \beta)$ , then  $\tau(x)(\alpha)$  only depends on the restriction of  $x$  to  $t \cdot M$ .  $\square$

**Proposition 6.2.** *Every cellular automaton  $\tau: Q^\Gamma \rightarrow Q^\Gamma$  is continuous.*

*Proof.* Let  $M$  be a memory set and  $(\alpha_0, T)$  a coordinate system for  $\tau$ . Let  $x \in Q^\Gamma$  and let  $W$  be a neighborhood of  $\tau(x)$  in  $Q^\Gamma$ . Then one can find a finite subset  $\Omega \subset \Gamma$  such that

$$V(\tau(x), \Omega) \subset W.$$

Consider the finite set  $\Omega M = \{t_\alpha \cdot \beta: \alpha \in \Omega, \beta \in M\}$ , where  $t_\alpha$  denotes the coordinate of  $\alpha$ . If  $y \in Q^\Gamma$  coincides with  $x$  on  $\Omega M$ , then  $\tau(x)$  and  $\tau(y)$  coincide on  $\Omega$  by Lemma 6.1. Thus we have

$$\tau(V(x, \Omega M)) \subset V(\tau(x), \Omega) \subset W.$$

This shows that  $\tau$  is continuous.  $\square$

**Lemma 6.3.** *Let  $\varphi: Q^\Gamma \rightarrow Q$  be a continuous map. Then there exists a finite subset  $M \subset \Gamma$  and a map  $\mu: Q^M \rightarrow Q$  such that  $\varphi(x) = \mu(x|_M)$  for all  $x \in Q^\Gamma$ .*

*Proof.* As the map  $\varphi: Q^\Gamma \rightarrow Q$  is continuous, we can find, for any  $x \in Q^\Gamma$ , a neighborhood  $W$  of  $x$  such that  $\varphi(W) = \{\varphi(x)\}$  and thus a finite subset  $\Omega_x \subset \Gamma$  such that  $V(x, \Omega_x) \subset W$ . The sets  $V(x, \Omega_x)$  form an open cover of  $Q^\Gamma$ . As  $Q$  is finite,  $Q^\Gamma$  is compact, and there is a finite subset  $F \subset Q^\Gamma$  such that the sets  $V(x, \Omega_x)$ ,  $x \in F$ , cover  $Q^\Gamma$ . Let us set  $M = \bigcup_{x \in F} \Omega_x$ . Then  $M$  is a finite subset of  $\Gamma$ .

Let  $x$  and  $y$  be two configurations in  $Q^\Gamma$  such that  $x$  and  $y$  coincide on  $M$ . There is a  $x_0 \in F$  such that  $x \in V(x_0, \Omega_{x_0})$ , i.e.,  $x$  and  $x_0$  coincide on  $\Omega_{x_0}$ . As  $x$  and  $y$  coincide on  $M \supset \Omega_{x_0}$ , we have  $y \in V(x_0, \Omega_{x_0})$ . Thus  $\varphi(x) = \varphi(y)$ , and there is a map  $\mu: Q^M \rightarrow Q$  such that  $\varphi(x) = \mu(x|_M)$  for all  $x \in Q^\Gamma$ .  $\square$

**Proposition 6.4.** *Let  $\tau: Q^\Gamma \rightarrow Q^\Gamma$  be a continuous map. If  $Eq(\tau)$  is a big subgroup of  $G$ , then  $\tau$  is a cellular automaton.*

*Proof.* Since  $Eq(\tau)$  is a big subgroup of  $G$ , there exists a coordinate system  $(\alpha_0, T)$  such that  $T \subset Eq(\tau)$ . As  $\tau$  is continuous, the map  $\varphi: Q^\Gamma \rightarrow Q$  defined by  $\varphi(x) = \tau(x)(\alpha_0)$  is continuous. From Lemma 6.3, there exists a finite subset  $M \subset \Gamma$  and a map  $\mu: Q^M \rightarrow Q$  such that  $\varphi(x) = \mu(x|_M)$  for all  $x \in Q^\Gamma$ . For any  $\alpha \in \Gamma$ , denote by  $t \in T$  the coordinate of  $\alpha$  in  $(\alpha_0, T)$ . One has

$$\tau(x)(\alpha) = \tau(x)(t \cdot \alpha_0) = t^{-1}\tau(x)(\alpha_0)$$

for all  $x \in Q^\Gamma$  and for all  $\alpha \in \Gamma$ . Then, since  $T \subset Eq(\tau)$ , we have

$$\tau(x)(\alpha) = \tau(t^{-1}x)(\alpha_0) = \varphi(t^{-1}x) = \mu(t^{-1}x|_M)$$

for all  $x \in Q^\Gamma$  and for all  $\alpha \in \Gamma$ . Therefore,  $\tau$  is a cellular automaton.  $\square$

**Corollary 6.5.** *Let  $\tau: Q^\Gamma \rightarrow Q^\Gamma$  be a continuous and  $H$ -equivariant map, where  $H$  is a big subgroup of  $G$ . Then  $\tau$  is a cellular automaton.*

Since  $G$  is a big subgroup of itself, we also have:

**Corollary 6.6.** *Let  $\tau: Q^\Gamma \rightarrow Q^\Gamma$  be a continuous and  $G$ -equivariant map. Then  $\tau$  is a cellular automaton.*

Let's recall the classical theorem of Hedlund, i.e., with  $\Gamma = G$  and  $G$  acting on itself by left multiplication. In this case, all the coordinate systems are  $(g, G)$  with  $g \in G$ . Thus a big subgroup of  $G$  is necessary  $G$  itself.

**Theorem.** (Hedlund, [4]) *A map  $\tau: Q^G \rightarrow Q^G$  is a cellular automaton if and only if  $\tau$  is a continuous map and  $Eq(\tau) = G$ .*

As a corollary to Propositions 6.2 and 6.4, we have a generalized version of Hedlund's theorem for equivariant cellular automata:

**Theorem 6.7.** *A map  $\tau: Q^\Gamma \rightarrow Q^\Gamma$  is an equivariant cellular automaton if and only if  $\tau$  is a continuous map and  $\text{Eq}(\tau)$  is a big subgroup of  $G$ .*

**Corollary 6.8.** *Let  $H$  be a big subgroup of  $G$ . A map  $\tau: Q^\Gamma \rightarrow Q^\Gamma$  is a  $H$ -equivariant cellular automaton if and only if  $\tau$  is a continuous map and  $H \subset \text{Eq}(\tau)$ .*

The  $G$ -equivariant cellular automata are characterized by the property that they admit all the coordinate systems:

**Proposition 6.9.** *Let  $\tau: Q^\Gamma \rightarrow Q^\Gamma$  be a cellular automaton. Then  $\tau$  is a  $G$ -equivariant cellular automaton if and only if any coordinate system on  $\Gamma$  is a coordinate system for  $\tau$ .*

*Proof.* Suppose first that  $\tau$  is  $G$ -equivariant, i.e.,  $G \subset \text{Eq}(\tau)$ . Let  $(\alpha_0, T)$  be a coordinate system on  $\Gamma$ . One has  $T \subset G \subset \text{Eq}(\tau)$ . Therefore, by Proposition 4.3, the pair  $(\alpha_0, T)$  is a coordinate system for  $\tau$ .

Conversely, suppose that any coordinate system on  $\Gamma$  is a coordinate system for  $\tau$ . Let  $(M, \mu, (\alpha_0, T))$  be a construction triple for  $\tau$ . By virtue of Proposition 3.8, it is enough to show that  $\mu$  is  $S$ -invariant, where  $S = \text{Stab}(\alpha_0)$  denotes the stabilizer subgroup of  $\alpha_0$  in  $G$ . Let  $s \in S$  and  $x \in Q^\Gamma$ , and let us show that  $\mu(sx|_M) = \mu(x|_M)$ . Pick a random cell  $\alpha_1 \in \Gamma \setminus \{\alpha_0\}$ , with coordinate  $t$  in  $(\alpha_0, T)$ . Since any coordinate system on  $\Gamma$  is a coordinate system for  $\tau$ , then  $(M, \mu, (\alpha_0, T'))$  is another construction triple for  $\tau$ , where

$$T' = (T \setminus \{t\}) \cup \{ts^{-1}\}.$$

Let us calculate  $\tau(tx)(\alpha_1)$ . In the coordinate system  $(\alpha_0, T)$ , we have

$$\tau(tx)(\alpha_1) = \mu((t^{-1}tx)|_M) = \mu(x|_M).$$

On the other hand, in the coordinate system  $(\alpha_0, T')$ , we have

$$\tau(tx)(\alpha_1) = \mu\left(\left((ts^{-1})^{-1}tx\right)|_M\right) = \mu(sx|_M).$$

Therefore one has  $\mu(x|_M) = \mu(sx|_M)$  for all  $s \in S$  and all  $x \in Q^\Gamma$ . Then  $\tau$  is a  $G$ -equivariant cellular automaton.  $\square$

**Definition 6.10.** One says that a cellular automaton  $\tau: Q^\Gamma \rightarrow Q^\Gamma$  is *reversible* if  $\tau$  is bijective and  $\tau^{-1}$  is also a cellular automaton.

**Lemma 6.11.** *For any bijective map  $\tau: Q^\Gamma \rightarrow Q^\Gamma$ , one has  $\text{Eq}(\tau^{-1}) = \text{Eq}(\tau)$ .*

*Proof.* For all  $g \in \text{Eq}(\tau)$ , we have

$$\tau^{-1}(gx) = \tau^{-1}(g\tau(\tau^{-1}(x))) = \tau^{-1}(\tau(g\tau^{-1}(x))) = g\tau^{-1}(x)$$

and then  $g \in \text{Eq}(\tau^{-1})$ . Therefore  $\text{Eq}(\tau) \subset \text{Eq}(\tau^{-1})$ . Applying the latter inclusion to  $\tau^{-1}$ , one has  $\text{Eq}(\tau^{-1}) \subset \text{Eq}((\tau^{-1})^{-1}) = \text{Eq}(\tau)$ . Thus  $\text{Eq}(\tau^{-1}) = \text{Eq}(\tau)$ .  $\square$

**Proposition 6.12.** *Let  $\tau: Q^\Gamma \rightarrow Q^\Gamma$  be an equivariant cellular automaton. Then the following conditions are equivalent:*

- (i) *the map  $\tau$  is bijective;*
- (ii) *the cellular automaton  $\tau$  is reversible.*

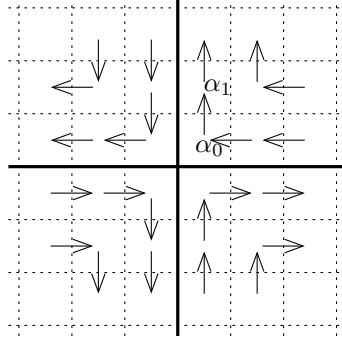
*Proof.* (ii) $\Rightarrow$ (i) is obvious. Conversely, suppose (i), i.e.,  $\tau$  is bijective. By Proposition 6.2,  $\tau$  is a continuous map. Since every continuous bijective map from a compact space to a Hausdorff space is a homeomorphism,  $\tau^{-1}$  is also continuous. As  $\tau$  is equivariant,  $\text{Eq}(\tau)$  is a big subgroup of  $G$ . Since  $\text{Eq}(\tau^{-1}) = \text{Eq}(\tau)$  by

Lemma 6.11,  $E_q(\tau^{-1})$  is a big subgroup of  $G$ . Finally, by Proposition 6.4,  $\tau^{-1}$  is a cellular automaton and then  $\tau$  is reversible.  $\square$

This proof shows moreover that a reversible equivariant cellular automaton can be reversed using the same coordinate system. This is not necessarily true for non-equivariant cellular automata, as in the following example.

**Example 6.13.** Consider Example 4.7. The map  $\tau$  is bijective. Let  $M' = \{\alpha_2\}$ , with  $\alpha_2$  the square of  $\Gamma$  whose center is  $(\frac{3}{2}, \frac{1}{2})$  and  $\mu': Q^{M'} \rightarrow Q$  defined by  $\mu(x) = x(\alpha_2)$  for all  $x \in Q^{M'}$ . Let  $T_2 = \{s_{(a,b)} \in G: a, b \in \mathbb{Z} \text{ and } 0 \leq b \leq |a|\}$  and  $T' = T_2 \cup rT_2 \cup r^2T_2 \cup r^3T_2$ . Consider the cellular automaton  $\tau'$  defined by the construction triple  $(M', \mu', (\alpha_0, T'))$ . Then one has  $\tau \circ \tau' = \tau' \circ \tau = \text{Id}_{Q^\Gamma}$  and hence  $\tau$  is reversible. Note that the pair  $(\alpha_0, T)$  is the only coordinate system for  $\tau$  and the pair  $(\alpha_0, T')$  is the only coordinate system for  $\tau^{-1}$ , up to the origin. Therefore the cellular automaton  $\tau$  is not reversible in its own coordinate system.

FIGURE 7. The reverse state shift automaton of Example 4.7



The arrows symbolize the displacement of the states by the action of  $\tau$ .

## 7. COMPOSITION OF CELLULAR AUTOMATA

Let  $\Gamma$  be a set equipped with a transitive left action of a group  $G$  and let  $Q$  be a nonempty finite set.

**Lemma 7.1.** *Let  $\tau: Q^\Gamma \rightarrow Q^\Gamma$  be a cellular automaton. Suppose there exists a cell  $\alpha_0 \in \Gamma$ , a finite subset  $M \subset \Gamma$  and a map  $\mu: Q^M \rightarrow Q$  such that  $\tau(x)(\alpha_0) = \mu(x|_M)$  for all  $x \in Q^\Gamma$ . Then  $(M, \mu, (\alpha_0, T))$  is a construction triple for  $\tau$  for some  $T \subset G$ .*

*Proof.* By Proposition 4.2, there exists a subset  $T \subset G$  such that the pair  $(\alpha_0, T)$  is a coordinate system for  $\tau$ . Then from (3.3) we have

$$\tau(x)(\alpha) = \tau(t^{-1}x)(\alpha_0) = \mu((t^{-1}x)|_M)$$

for all  $x \in Q^\Gamma$  and  $\alpha \in \Gamma$ , where  $t \in T$  denotes the coordinate of  $\alpha$ . Thus  $(M, \mu, (\alpha_0, T))$  is a construction triple for  $\tau$ .  $\square$

Let  $\tau_1$  and  $\tau_2: Q^\Gamma \rightarrow Q^\Gamma$  be cellular automata with construction triples  $(M_1, \mu_1, (\alpha_1, T_1))$  and  $(M_2, \mu_2, (\alpha_2, T_2))$  respectively. We construct a cellular automaton  $\tau'$  with the construction triple  $(M, \mu, (\alpha_1, T_1))$  defined this way: let

$$M = \{t_{\beta_1} \cdot \beta_2: \beta_1 \in M_1 \text{ and } \beta_2 \in M_2\}$$

where  $t_{\beta_1}$  denotes the coordinate of  $\beta_1$  in the coordinate system  $(\alpha_2, T_2)$ ; for  $y \in Q^M$  and  $t \in T_2$  the coordinate of an element of  $M_1$  in the coordinate system  $(\alpha_2, T_2)$ ,



define  $y_t: M_2 \rightarrow Q$  by setting  $y_t(\alpha) = y(t \cdot \alpha)$  for all  $\alpha \in M_2$ . Also, let  $\bar{y}: M_1 \rightarrow Q$  be the map defined by  $\bar{y}(\alpha) = \mu_2(y_t)$  for all  $\alpha \in M_1$  with coordinate  $t \in T_2$ . Finally define the map  $\mu: Q^M \rightarrow Q$  by setting

$$\mu(y) = \mu_1(\bar{y})$$

for all  $y \in Q^M$ . Then we have the following proposition:

**Proposition 7.2.** *With the above notation, if the composite map  $\tau = \tau_1 \circ \tau_2$  is a cellular automaton, then  $\tau = \tau'$ .*

*Proof.* From Lemma 7.1 it is sufficient to prove that  $\tau(x)(\alpha_1) = \mu(x|_M)$  for all  $x \in Q^\Gamma$ . Let  $x \in Q^\Gamma$  be a configuration and  $\beta_1 \in M_1$  (resp.  $\beta_2 \in M_2$ ) be a cell with coordinate  $t_1 \in T_2$  (resp.  $t_2 \in T_2$ ). We have

$$(t_1^{-1}x)(\beta_2) = x(t_1 \cdot \beta_2) = x|_M(t_1 \cdot \beta_2) = (x|_M)_{t_1}(\beta_2)$$

and thus  $t_1^{-1}x|_{M_2} = (x|_M)_{t_1}$ . Therefore one has

$$\tau_2(x)(\beta_1) = \mu_2(t_1^{-1}x|_{M_2}) = \mu_2((x|_M)_{t_1}) = \overline{x|_M}(\beta_1)$$

and thus  $\tau_2(x)|_{M_1} = \overline{x|_M}$ . Finally we have

$$\tau_1 \circ \tau_2(x)(\alpha_1) = \mu_1(\tau_2(x)|_{M_1}) = \mu_1(\overline{x|_M}) = \mu(x|_M)$$

and thus  $\tau = \tau_1 \circ \tau_2 = \tau'$ .  $\square$

Note that it may happen that  $\tau_1 \circ \tau_2$  is not a cellular automaton. The following proposition gives a sufficient condition for  $\tau_1 \circ \tau_2$  to be a cellular automaton, when  $\tau_1$  and  $\tau_2$  are equivariant cellular automata. Note that the intersection of two big subgroups may not be a big subgroup.

**Proposition 7.3.** *Let  $\tau_1$  and  $\tau_2: Q^\Gamma \rightarrow Q^\Gamma$  be cellular automata. If  $Eq(\tau_1) \cap Eq(\tau_2)$  is a big subgroup of  $G$ , then  $\tau_1 \circ \tau_2$  is a cellular automaton.*

*Proof.* From Proposition 6.2,  $\tau_1$  and  $\tau_2$  are continuous maps, therefore  $\tau_1 \circ \tau_2$  is a continuous map. As  $Eq(\tau_1) \cap Eq(\tau_2) \subset Eq(\tau_1 \circ \tau_2)$ ,  $Eq(\tau_1 \circ \tau_2)$  is also big subgroup of  $G$ . Thus, by Proposition 6.4,  $\tau_1 \circ \tau_2$  is a cellular automaton.  $\square$

From Proposition 7.3, we deduce that if  $\tau$  is an equivariant cellular automaton, then  $\tau \circ \tau$  is also a cellular automaton. But it may happen that  $\tau \circ \tau$  is not a cellular automaton, if  $\tau$  is not an equivariant cellular automaton.

**Example 7.4.** (a) **The Margolus billiard-ball cellular automaton.** Consider the cellular automata  $\tau_0$  and  $\tau_1$  and the map  $\tau = \tau_1 \circ \tau_0$  defined in Example 3.3 (e). Remark that  $T_0$  is a subgroup of  $G$  and that  $T_0 = t_0 T_0 t_0^{-1}$  and therefore  $\tau_0$  and  $\tau_1$  are  $T_0$ -equivariant. Hence by Proposition 7.3 the Margolus billiard-ball  $\tau$  is a cellular automaton. As  $\tau_0$  and  $\tau_1$  are reversible,  $\tau$  is bijective. Then, since  $\tau$  is equivariant,  $\tau$  is reversible by Proposition 6.12. The Margolus billiard-ball is an important example since Margolus in [10] proved that it is a universal cellular automaton. Still, there was no formal proof that  $\tau_1 \circ \tau_0$  was a cellular automaton. Indeed, it may happen that the composition of two cellular automata is no longer a cellular automaton, as one can see in the following example.

(b) Consider the cellular automaton defined in Example 3.3 (d). We construct the cellular automaton  $\tau'$  as in Proposition 7.2 with  $\tau_1 = \tau_2 = \tau$ . Then we have  $M' = \{\alpha_3\}$  with  $\alpha_3$  the square of  $\Gamma$  whose center is  $(-\frac{1}{2}, \frac{3}{2})$ ,  $\mu': Q^{M'} \rightarrow Q$  defined by  $\mu'(x) = x(\alpha_3)$  for all  $x \in Q^M$ , and the construction triple  $(M', \mu', (\alpha_0, T))$  defines a cellular automaton  $\tau'$ . By Proposition 7.2, we know that if  $\tau \circ \tau$  is a

cellular automaton, then  $\tau \circ \tau = \tau'$ . Let  $\beta_1$  (resp.  $\beta_2, \beta_3$ ) denote the square of  $\Gamma$  whose center is  $(\frac{3}{2}, \frac{1}{2})$  (resp.  $(\frac{3}{2}, \frac{5}{2}), (\frac{1}{2}, \frac{3}{2})$ ). Then one has for all  $x \in Q^\Gamma$ ,

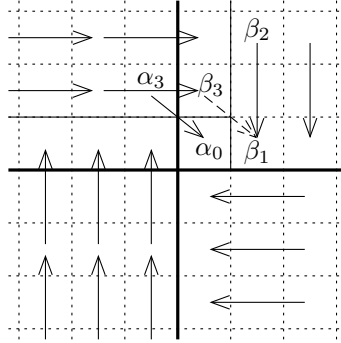
$$\tau \circ \tau(x)(\beta_1) = x(\beta_2)$$

and

$$\tau'(x)(\beta_1) = x(\beta_3)$$

Hence  $\tau \circ \tau$  is not a cellular automaton. Note that this also proves that  $\tau$  is not an equivariant cellular automaton.

FIGURE 8. Comparison between  $\tau \circ \tau$  and  $\tau'$



The plain arrows symbolize the displacement of states by the action of  $\tau \circ \tau$  and the dash arrow symbolizes the displacement of a state by the action of  $\tau'$ .

Denote by  $\mathcal{CA}(\Gamma, Q)$  the set of cellular automata over the state set  $Q$  and the universe  $\Gamma$ . The latter example shows that  $\mathcal{CA}(\Gamma, Q)$  is not stable for the composition of maps, and any subset of  $\mathcal{CA}(\Gamma, Q)$  containing the cellular automaton of Example 7.4 (b) is not stable either. But there are subsets of  $\mathcal{CA}(\Gamma, Q)$  which are stable for the composition of maps: for every coordinate system  $(\alpha_0.T)$ , denote by  $\mathcal{CA}(\Gamma, Q, (\alpha_0.T))$  the subset of  $\mathcal{CA}(\Gamma, Q)$  of cellular automata  $\tau$  such that  $T \subset Eq(\tau)$ . As a corollary to Proposition 7.3, we have the following:

**Corollary 7.5.** *For every coordinate system  $(\alpha_0.T)$ , the set  $\mathcal{CA}(\Gamma, Q, (\alpha_0.T))$  is a monoid for the composition of maps.*

For every big subgroup  $H$  of  $G$ , denote by  $\mathcal{CA}(\Gamma, Q, H)$  the subset of  $\mathcal{CA}(\Gamma, Q)$  of cellular automata  $\tau$  such that  $H \subset Eq(\tau)$ . As a corollary to Proposition 7.3, we have the following:

**Corollary 7.6.** *For every big subgroup  $H$  of  $G$  and every coordinate system  $(\alpha_0.T)$  such that  $T \subset H$ , the set  $\mathcal{CA}(\Gamma, Q, H)$  is a submonoid of  $\mathcal{CA}(\Gamma, Q, (\alpha_0.T))$ . The set  $\mathcal{CA}(\Gamma, Q, G)$  is a submonoid of  $\mathcal{CA}(\Gamma, Q, H)$  for every big subgroup  $H$ .*

## 8. CONCLUSION

The question arises whether other classical theorems on cellular automata are also true for  $G$ -set cellular automata. As an example, we can take the Garden of Eden theorem, characterizing surjective cellular automata as pre-injective cellular automata. As the equivalence between reversibility and bijectivity has been proven for equivariant cellular automaton, another natural question is: does there exist a non-equivariant non-reversible bijective cellular automaton?

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